5.7 DESIGN OF HIGH-RESOLUTION QUADRATIC TFDs WITH SEPARABLE KERNELS [0]

5.7.1 RIDs and Quadratic TFDs

Reduced-interference distributions (RIDs) smooth out the unwanted cross-terms in the time-frequency (t, f) domain [see Chapter 3 and Article 5.2]. The spectrogram is the best-known RID, but not the only one; it is well known that a time-frequency distribution (TFD) is a RID if its kernel has a two-dimensional low-pass characteristic in the Doppler-lag (ν, τ) domain [1]. A suitably chosen **separable kernel**, i.e. a kernel with the form $G_1(\nu)\, g_2(\tau)$, can meet this requirement while giving higher time-frequency resolution than the spectrogram; for example, the "smoothed WVD" discussed in [2] has a separable kernel. It has also been shown that if the kernel is *independent of lag* (i.e. a function of Doppler alone in the Doppler-lag domain, or of time alone in the time-lag domain), then the resulting TFD can exhibit fine frequency resolution and high attenuation of cross-terms [3, 4].

This article explores the properties of TFDs with separable kernels, including lag-independent kernels, characterizes the signals for which such kernels can be recommended, and gives examples of RID designs using such kernels. It builds on the argument presented in Sections 3.2.1 and 3.2.2 (pp. 66–69).

5.7.2 Separable Kernel Formulations

In the Doppler-lag domain, a **separable kernel** has the form

$$g(\nu, \tau) = G_1(\nu)\, g_2(\tau). \tag{5.7.1}$$

If we let

$$G_1(\nu) = \underset{t \to \nu}{\mathcal{F}}\{g_1(t)\} \tag{5.7.2}$$

$$G_2(f) = \underset{\tau \to f}{\mathcal{F}}\{g_2(\tau)\}, \tag{5.7.3}$$

then the relationships shown in the graphical Eq. (3.2.6) become

$$\begin{tikzcd}
 & g_1(t)\, G_2(f) \arrow[dr, "t \to \nu"] & \\
g_1(t)\, g_2(\tau) \arrow[ur, "\tau \to f"] \arrow[dr, "t \to \nu"'] & & G_1(\nu)\, G_2(f). \\
 & G_1(\nu)\, g_2(\tau) \arrow[ur, "\tau \to f"'] &
\end{tikzcd} \tag{5.7.4}$$

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Substituting Eq. (5.7.1) into Eq. (3.2.11), we obtain the filtered ambiguity function

$$\mathcal{A}_z(\nu,\tau) = G_1(\nu)\, g_2(\tau)\, A_z(\nu,\tau) \tag{5.7.5}$$

(also called the "generalized ambiguity function" by some authors). Then, using the convolution properties and the notations of the graphical Eq. (3.2.10), we find

$$\begin{aligned} r_z(\nu,f) &= G_1(\nu)\, G_2(f) \underset{f}{*} k_z(\nu,f) && (5.7.6)\\ R_z(t,\tau) &= g_2(\tau)\, g_1(t) \underset{t}{*} K_z(t,\tau) && (5.7.7)\\ \rho_z(t,f) &= g_1(t) \underset{t}{*} W_z(t,f) \underset{f}{*} G_2(f) && (5.7.8) \end{aligned}$$

where $K_z(t,\tau) = z(t{+}\frac{\tau}{2})\, z^*(t{-}\frac{\tau}{2})$ is the (unsmoothed) IAF, $k_z(\nu,f)$ is its 2DFT (the spectral autocorrelation function), $W_z(t,f)$ is the WVD, $r_z(\nu,f)$ is the smoothed spectral autocorrelation function, $R_z(t,\tau)$ is the smoothed IAF, and $\rho_z(t,f)$ is the quadratic TFD.

Eq. (5.7.7) shows that the effect of the lag-dependent factor on the TFD is simply "lag windowing", i.e. multiplication by the same factor in the (t,τ) domain before transforming to the (t,f) domain. In Eq. (5.7.8), the two convolutions are associative (i.e. can be performed in either order), so that we may consider the Doppler-dependent and lag-dependent factors as leading to separate convolutions in time and frequency, respectively.

A **Doppler-independent** (DI) kernel is a special case of a separable kernel obtained by putting

$$G_1(\nu) = 1 \tag{5.7.9}$$

in Eqs. (5.7.1) and (5.7.2), which then become

$$\begin{aligned} g(\nu,\tau) &= g_2(\tau) && (5.7.10)\\ g_1(t) &= \delta(t). && (5.7.11) \end{aligned}$$

Making these substitutions in Eqs. (5.7.4) to (5.7.8), we obtain

$$\begin{tikzcd} & \delta(t)\, G_2(f) \arrow[dr, "t", "\nu"'] & \\ \delta(t)\, g_2(\tau) \arrow[ur, "\tau", "f"'] \arrow[dr, "t"', "\nu"] & & G_2(f) \\ & g_2(\tau) \arrow[ur, "\tau", "f"'] & \end{tikzcd} \tag{5.7.12}$$

$$\begin{aligned} \mathcal{A}_z(\nu,\tau) &= g_2(\tau)\, A_z(\nu,\tau) && (5.7.13)\\ r_z(\nu,f) &= G_2(f) \underset{f}{*} k_z(\nu,f) && (5.7.14)\\ R_z(t,\tau) &= g_2(\tau)\, K_z(t,\tau) && (5.7.15)\\ \rho_z(t,f) &= G_2(f) \underset{f}{*} W_z(t,f). && (5.7.16) \end{aligned}$$

As seen in Eq. (5.7.12), a "Doppler-independent" kernel is indeed independent of Doppler in all four domains. In the Doppler-lag domain it is a function of lag alone. Eq. (5.7.16) shows that a DI kernel causes smoothing (or "smearing") of the WVD in the frequency direction only. The graphical Eq. (3.2.10) defined the notation

$$\rho_z(t,f) = \underset{\tau\to f}{\mathcal{F}}\{R_z(t,\tau)\}. \tag{5.7.17}$$

Substituting Eq. (5.7.15) into Eq. (5.7.17) gives

$$\rho_z(t,f) = \underset{\tau\to f}{\mathcal{F}}\{g_2(\tau)\, K_z(t,\tau)\} \tag{5.7.18}$$

which shows that a quadratic TFD with a DI kernel is a *windowed WVD*; the "windowing" is applied in the lag direction before Fourier transformation from lag to frequency.

A **lag-independent** (LI) kernel is another special case of a separable kernel, obtained by putting

$$g_2(\tau) = 1 \tag{5.7.19}$$

in Eqs. (5.7.1) and (5.7.3), which then become

$$g(\nu,\tau) \;=\; G_1(\nu) \tag{5.7.20}$$
$$G_2(f) \;=\; \delta(f). \tag{5.7.21}$$

Making these substitutions in Eqs. (5.7.4) to (5.7.8), we obtain

$$\begin{tikzcd}
 & g_1(t)\,\delta(f) \arrow[dr, "t", "\nu"'] & \\
g_1(t) \arrow[ur, "f"', "\tau"] \arrow[dr, "t", "\nu"'] & & G_1(\nu)\,\delta(f) \\
 & G_1(\nu) \arrow[ur, "f"', "\tau"] &
\end{tikzcd} \tag{5.7.22}$$

$$\mathcal{A}_z(\nu,\tau) \;=\; G_1(\nu)\, A_z(\nu,\tau) \tag{5.7.23}$$
$$r_z(\nu,f) \;=\; G_1(\nu)\, k_z(\nu,f) \tag{5.7.24}$$
$$R_z(t,\tau) \;=\; g_1(t) \underset{t}{*} K_z(t,\tau) \tag{5.7.25}$$
$$\rho_z(t,f) \;=\; g_1(t) \underset{t}{*} W_z(t,f). \tag{5.7.26}$$

The last result shows that an LI kernel causes smoothing of the WVD in the time direction only.

As seen in Eq. (5.7.22), a "lag-independent" kernel is indeed independent of lag in all four domains. In the time-lag domain it is a function of time alone; for this reason, such kernels have been called "time-only kernels" [3, 4].

The WVD kernel is $g(\nu,\tau) = 1$, which is both Doppler-independent and lag-independent; it may be regarded as DI with $g_2(\tau) = 1$ or as LI with $G_1(\nu) = 1$.

Table 5.7.1: TFD properties and associated kernel requirements for separable, Doppler-independent and lag-independent kernels. *Explanation of properties*: **Time marginal:** The integral of the TFD over frequency is the instantaneous power. **Freq. marginal:** The integral of the TFD over time is the energy spectrum. **Inst. Freq.:** The IF is the first moment of the TFD w.r.t. frequency. **Time delay:** The time delay is the first moment of the TFD w.r.t. time. **Time support:** If the non-zero values of the signal are confined to a certain time interval, so are the non-zero values of the TFD. **Freq. support:** If the non-zero values of the spectrum are confined to a certain frequency range, so are the non-zero values of the TFD. "WVD only*" (with asterisk) means a WVD multiplied by an arbitrary constant.

PROPERTY	KERNEL CONSTRAINTS		
	Separable $g(\nu,\tau)=G_1(\nu)\,g_2(\tau)$	DI $G_1(\nu)=1$	LI $g_2(\tau)=1$
Realness	$G_1(\nu)\,g_2(\tau)$ $=G_1^*(-\nu)\,g_2^*(-\tau)$.	$G_2(f)$ is real.	$g_1(t)$ is real.
Time marginal	$G_1(\nu)\,g_2(0)=1\ \ \forall\nu$	$g_2(0)=1$	WVD only
Freq. marginal	$G_1(0)\,g_2(\tau)=1\ \ \forall\tau$	WVD only	$G_1(0)=1$
Inst. freq.	$G_1(\nu)\,g_2(0)=\text{const.}$ $g_2'(0)=0$.	$g_2'(0)=0$	WVD only*
Time delay	$G_1(0)\,g_2(\tau)=\text{const.}$ $G_1'(0)=0$.	WVD only*	$G_1'(0)=0$
Time support	DI only	Always	WVD only*
Freq. support	LI only	WVD only*	Always
RID potential	Unrestricted	Inner artifacts	Cross-terms

5.7.3 Properties

Table 5.7.1 is extracted from Table 3.3.1 on p. 75. The properties of time-shift invariance and frequency-shift invariance are omitted, being common to all quadratic TFDs. Positivity is omitted because, in practice, the design of a separable kernel involves a deliberate sacrifice of non-negativity in favor of higher (t,f) resolution.

Let us define a **proper** DI or LI kernel as one that is *non*-constant (so that the resulting TFD is *not* a WVD, with or without amplitude scaling). Similarly, let us define a **proper** separable kernel as one that is neither DI nor LI. Then Table 5.7.1 shows that a TFD with a DI kernel can satisfy the realness, time marginal, time support and instantaneous frequency (IF) properties; but no proper DI kernel satisfies the frequency marginal, frequency support or time delay property. Similarly, a TFD with an LI kernel can satisfy the realness, frequency marginal, frequency support and time delay properties; but no proper LI kernel satisfies the time marginal, time support or IF property.

The reduced-interference property ("RID potential") requires further explanation. The WVD may contain interference terms of two kinds. **Inner artifacts** or "inner interference terms" [see Article 4.2] are caused by nonlinear frequency modulation laws, and cause the WVD to alternate as we move normal to the expected feature(s) in the (t, f) plane. In the case of a multicomponent signal, **Cross-terms** or "outer interference terms" [see Article 4.2] are caused by cross-product terms in the IAF $K_z(t, \tau)$, and cause the WVD to alternate as we move parallel to the expected features in the (t, f) plane.

Thus the inner artifacts alternate as we move in the frequency direction, and may therefore be suppressed by convolution with a sufficiently long $G_2(f)$ [see Eq. (5.7.16)], which corresponds to a sufficiently short $g_2(\tau)$. This is possible for a DI kernel but not an LI kernel. Similarly, the cross-terms alternate as we move in the time direction, and may therefore be suppressed by convolution with a sufficiently long $g_1(t)$ [see Eq. (5.7.26)], which corresponds to a sufficiently short $G_1(\nu)$. This is possible for an LI kernel but not a DI kernel. A proper separable kernel causes convolution in both time and frequency and can suppress both kinds of interference terms.

For an LI kernel, the suppression of cross-terms is facilitated if the components are of slowly-varying frequencies, so that the cross-terms extend (and alternate) approximately in the time direction. Furthermore, the loss of frequency resolution caused by convolution with $g_1(t)$ is proportional to the rate of change of the IF; for constant frequency, the components run parallel to the time axis, so that there is no loss of resolution apart from that caused by the time-variation of frequency resolution in the WVD.

5.7.4 Design Examples of Separable-Kernel TFDs

Early experience suggested that the kernel of a RID must exhibit a *two-dimensional* low-pass characteristic in the Doppler-lag domain (see e.g. [1], pp. 79–81). The **B-distribution** (**BD**) defined in [5] has the separable time-lag kernel

$$G_{\mathrm{B}}(t, \tau) = |\tau|^{\beta} \cosh^{-2\beta} t \qquad (5.7.27)$$

where β is a positive real parameter that controls the degree of smoothing. This kernel is low-pass in the Doppler dimension but *not* in the lag dimension. But, paradoxically, the B-distribution has shown impressive reduced-interference properties for certain signals [5].

The best results from the BD are consistently obtained for small positive values of β, for which the lag-dependent factor is nearly constant apart from a "slot" at $\tau = 0$. Moreover, any desired lag-dependence can be introduced later by windowing prior to Fourier transformation from τ to f, as is often done for computational economy or improved time resolution. Accordingly, the BD was modified in [3, 4] by making the "lag-dependent" factor *exactly* constant. The resulting **modified B-distribution** (**MBD**) had an LI kernel. This inspired a more thorough inquiry into the properties of LI kernels, and the findings explained the behavior of the BD,

whose kernel may fairly be described as "nearly LI" for small positive β.

The time-lag kernel of the MBD is

$$G_{\mathrm{MB}}(t,\tau) = g_{\beta}(t) = \frac{\cosh^{-2\beta} t}{\int_{-\infty}^{\infty} \cosh^{-2\beta}\xi \, d\xi} \tag{5.7.28}$$

where β is a positive real parameter and the denominator is for normalization. The graph of $g_{\beta}(t)$ vs. t is a bell-shaped curve whose spread is inversely related to β.

If the signal is a tone or sum of tones, we can obtain closed-form expressions showing that the MBD has optimal concentration about the IF law. In particular, the MBD of a single tone is a delta function of frequency, while the MBD of the sum of two tones comprises two delta functions plus a cross-term whose amplitude is controlled by β (see [6] and Article 10.3). For most signals there is no closed-form expression for the MBD, so we must resort to numerical computations with discretized variables. For discrete time n and discrete lag m, the MBD kernel becomes

$$G_{\mathrm{MB}}(n,m) = g_{\beta}(n) = \frac{\cosh^{-2\beta} n}{\sum_i \cosh^{-2\beta} i}. \tag{5.7.29}$$

The following numerical examples include one MBD for comparison. They also include TFDs with separable kernels using two types of window functions that are well known in digital signal processing and spectral analysis. The M-point **Hamming** function, where M is odd, is

$$\mathrm{hamm}_M(i) = 0.54 + 0.46 \cos \tfrac{2\pi i}{M} \ ; \quad -\tfrac{M-1}{2} \leq i \leq \tfrac{M-1}{2} \tag{5.7.30}$$

where i is discrete time or lag. The L-point **Hanning** function, where L is odd, is

$$\mathrm{hann}_L(i) = 0.5 + 0.5 \cos \tfrac{2\pi i}{L} \ ; \quad -\tfrac{L-1}{2} \leq i \leq \tfrac{L-1}{2}. \tag{5.7.31}$$

In the numerical examples given below, the Hamming function is preferred in the lag direction and the Hanning in the time direction, in order to minimize ripples in the (t, f) domain; and whenever a proper separable kernel is compared with DI and LI kernels, it has the same lag factor as the DI kernel and the same Doppler factor (or time factor) as the LI kernel.

5.7.5 Results and Discussion

Some numerical computations of separable-kernel TFDs, and of other TFDs for purposes of comparison, are presented in the following graphs. Complete specifications of signals and kernels are given in the figure captions for ease of reference. Each graph includes the TFD (main panel, with time axis vertical), time plot (left panel) and magnitude spectrum (bottom panel).

Fig. 5.7.1 shows two TFDs of a signal comprising a tone (constant frequency) and a chirp (linearly increasing frequency). Part (a) shows the WVD, with the prominent cross-term. Part (b) shows the effect of an LI kernel, which smoothes

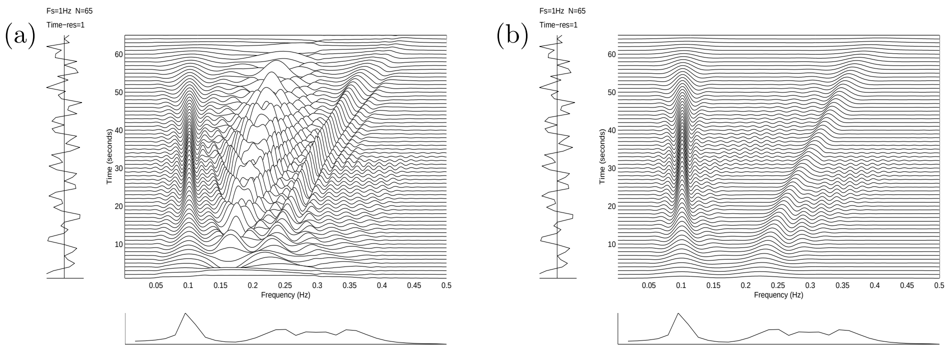

Fig. 5.7.1: TFDs of the sum of a tone (frequency 0.1) and a linear FM signal (frequency range 0.2 to 0.4), unit amplitudes, duration 65 samples, sampling rate 1 Hz: (a) WVD; (b) lag-independent, $G(n,m) = \mathrm{hann}_{15}(n)$.

the WVD in the time direction, suppressing the oscillatory cross-term. For the tone, this smoothing causes only a slight loss of frequency resolution, and only because the frequency resolution of the WVD varies with time. For the chirp, the loss of frequency resolution is greater because the direction of smoothing is not parallel to the IF law; that is, the smoothing is "across" as well as "along" the component.

Fig. 5.7.2 compares six TFDs of a two-component signal with slowly-varying frequencies; the lower-frequency component is a pure tone. In Fig. 5.7.3, the tone is replaced by a faster-varying sinusoidal FM signal (nonlinear FM). In each figure, part (a) shows the WVD, while part (b) shows the effect of a DI kernel, part (c) an LI kernel, and part (d) a separable kernel combining the lag and time functions of parts (b) and (c). Note that (b) and (d) are related by the same time-smoothing as (a) and (c), while (c) and (d) are related by the same frequency-smoothing as (a) and (b). In each figure, part (f) shows a spectrogram for comparison with the separable-kernel TFD.

In Fig. 5.7.2, a cross-term is prominent in the WVD (a) and is not suppressed by the DI kernel (b). It is suppressed by the Hanning LI kernel (c) and the MBD kernel (e), which is also LI. Both LI kernels are bell-shaped functions of time, and their parameters have been chosen to give similar degrees of smoothing in time, accounting for the similarity between graphs (c) and (e). The proper separable kernel (d) has lower frequency resolution, but less ripple about the IF laws, than the LI kernels. The spectrogram (f) gives the best suppression of artifacts and the lowest resolution. Comparing Fig. 5.7.1(b) and Fig. 5.7.2(c), we see that the faster-varying IF, which causes a faster-varying cross-term frequency and lower minimum beat frequency, needs a longer time-smoothing function for adequate suppression of cross-terms.

In Fig. 5.7.3, both cross-terms and inner artifacts are visible. The cross-terms appear as "rough terrain" between the components, while the inner artifacts appear as spurious ridges in the sinusoidal FM component; both types are prominent in the WVD (a). The DI kernel (b) is effective against the inner artifacts. The LI

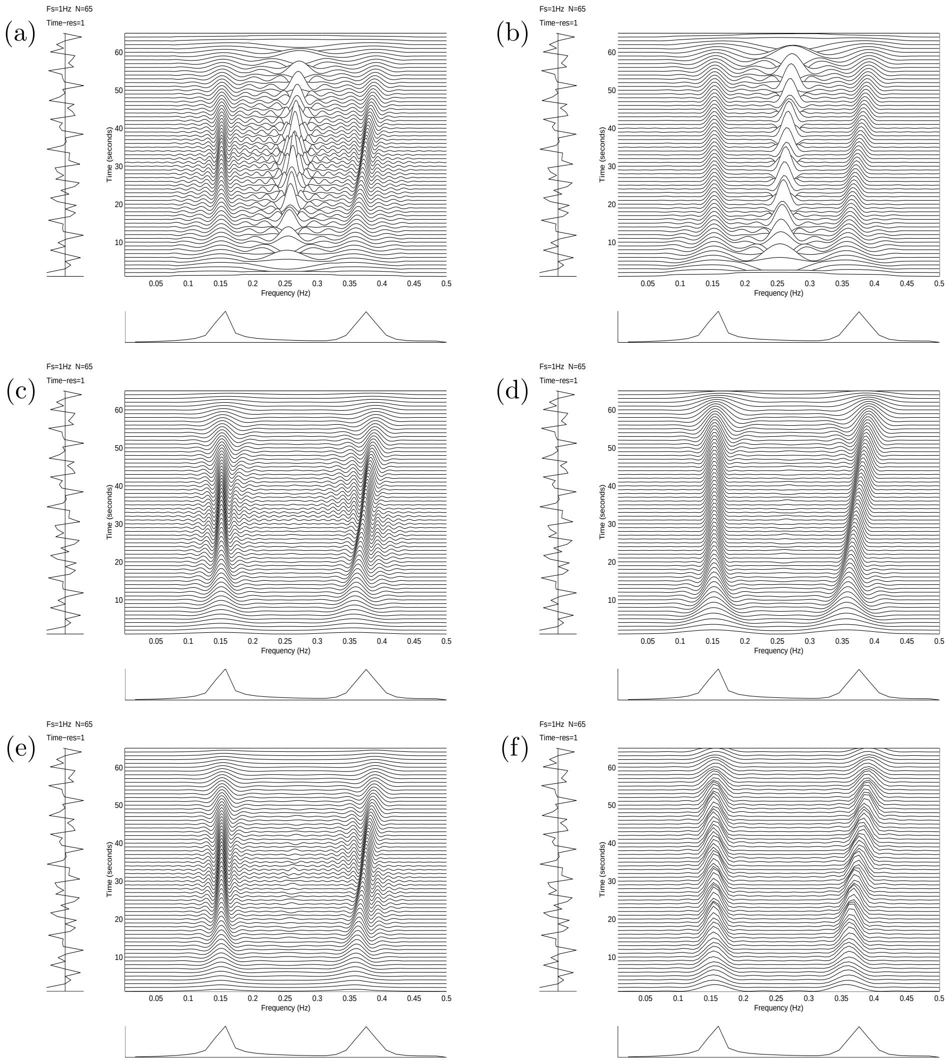

Fig. 5.7.2: TFDs of the sum of a tone (frequency 0.15) and a linear FM signal (frequency range 0.35 to 0.4), unit amplitudes, duration 65 samples, sampling rate 1 Hz: (a) WVD; (b) Doppler-independent, $G(n,m) = \mathrm{hamm}_{47}(m)$; (c) lag-independent, $G(n,m) = \mathrm{hann}_{11}(n)$; (d) separable, $G(n,m) = \mathrm{hann}_{11}(n)\ \mathrm{hamm}_{47}(m)$; (e) modified B, $\beta = 0.2$; (f) spectrogram, 35-point rectangular window.

kernel (c) is effective against the cross-terms. The separable kernel (d) is effective against both. The well-known Choi-Williams distribution (e), with the parameter σ visually optimized, spreads the inner artifacts in time and the cross-terms in frequency but does not satisfactorily suppress either. In this example the separable-kernel TFD (d) appears to offer a better compromise between resolution and cross-

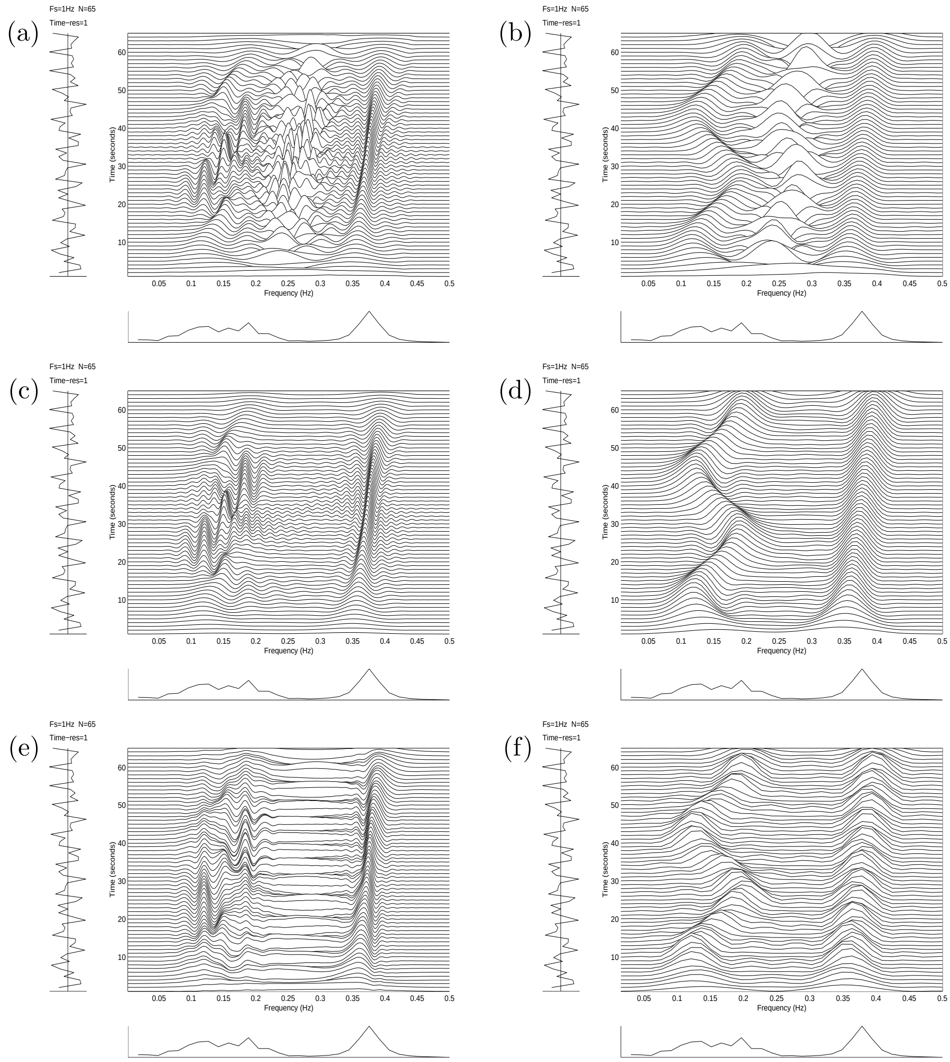

Fig. 5.7.3: TFDs of the sum of a sinusoidal FM signal (frequency 0.15±0.05, 2 cycles of modulation) and a linear FM signal (frequency range 0.35 to 0.4), unit amplitudes, duration 65 samples, sampling rate 1 Hz: (a) WVD; (b) Doppler-independent, $G(n,m) = \text{hamm}_{23}(m)$; (c) lag-independent, $G(n,m) = \text{hann}_{11}(n)$; (d) separable, $G(n,m) = \text{hann}_{11}(n)\ \text{hamm}_{23}(m)$; (e) Choi-Williams, $\sigma = 1$; (f) spectrogram, 17-point rectangular window.

term suppression than either the spectrogram (f) or the CWD (e), neither of which has a separable kernel. Comparing Fig. 5.7.2(d) with Fig. 5.7.3(d), we may confirm that a more nonlinear IF requires a shorter lag window (for the suppression of inner artifacts), giving coarser frequency resolution.

5.7.6 Summary and Conclusions

A separable kernel gives separate control of the frequency-smoothing and time-smoothing of the WVD: the lag-dependent factor causes a convolution in the frequency direction in the (t, f) plane, while the Doppler-dependent factor causes a convolution in the time direction. A Doppler-*in*dependent (DI) kernel smoothes the WVD in the frequency direction only, reducing the inner artifacts and preserving the time marginal. A lag-*in*dependent (LI) kernel smoothes the WVD in the time direction only, reducing the cross-terms and preserving the frequency marginal.

For an LI kernel, slower variations in component frequencies allow easier suppression of cross-terms and higher frequency resolution in the TFD; such kernels should therefore be considered for representing multicomponent signals with slowly-varying instantaneous frequencies. For a multicomponent signal with at least one highly nonlinear IF law, a lag-dependent factor is also needed to suppress the inner artifacts.

The separable-kernel approach allows a complete understanding and appraisal of the properties and behavior of the smoothed WVD [2], the B-distribution [5] and the modified B-distribution [3,4]. It also enables the construction of high-resolution quadratic TFDs using the classical smoothing functions commonly encountered in digital signal processing and spectral analysis.

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